On sum edge-coloring of regular, bipartite and split graphs

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An edge-coloring of a graph G with natural numbers is called a sum edge-coloring if the colors of edges incident to any vertex of G are distinct and the sum of the colors of the edges of G is minimum. The edge-chromatic sum of a graph G is the sum of the colors of edges in a sum edge-coloring of G. It is known that the problem of finding the edge-chromatic sum of an r-regular ($r \geq 3$) graph is NP-complete. In this paper we give a polynomial time $\left(1+\frac{2r}{(r+1)^2}\right)$-approximation algorithm for the edge-chromatic sum problem on r-regular graphs for $r \geq 3$. Also, it is known that the problem of finding the edge-chromatic sum of bipartite graphs with maximum degree 3 is NP-complete. We show that the problem remains NP-complete even for some restricted class of bipartite graphs with maximum degree 3. Finally, we give upper bounds for the edge-chromatic sum of some split graphs.

Keywords: edge-coloring, sum edge-coloring, regular graph, bipartite graph, split graph

1. Introduction

We consider finite undirected graphs that do not contain loops or multiple edges. Let $V(G)$ and $E(G)$ denote sets of vertices and edges of G, respectively. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, that is, $V(G[S]) = S$ and $E(G[S])$ consists of those edges of $E(G)$ for which both ends are in S. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of G by $\Delta(G)$, the chromatic number of G by $\chi(G)$, and the chromatic index of G by $\chi'(G)$. The terms and concepts that we do not define can be found in [4, 26].

A proper vertex-coloring of a graph G is a mapping $\alpha : V(G) \to \mathbf{N}$ such that $\alpha(u) \neq \alpha(v)$ for every $uv \in E(G)$. If α is a proper vertex-coloring of a graph G, then $\Sigma(G, \alpha)$

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denotes the sum of the colors of the vertices of G. For a graph G, define the vertex-chromatic sum $\Sigma(G)$ as follows: $\Sigma(G) = \min_\alpha \Sigma(G,\alpha)$, where minimum is taken among all possible proper vertex-colorings of G. If α is a proper vertex-coloring of a graph G and $\Sigma(G) = \Sigma(G,\alpha)$, then α is called a sum vertex-coloring. The strength of a graph G ($s(G)$) is the minimum number of colors needed for a sum vertex-coloring of G. The concept of sum vertex-coloring and vertex-chromatic sum was introduced by Kubicka [16] and Supowit [22]. In [17], Kubicka and Schwenk showed that the problem of finding the vertex-chromatic sum is NP-complete in general and polynomial time solvable for trees. Jansen [12] gave a dynamic programming algorithm for partial k-trees. In papers [5, 6, 10, 13, 18], some approximation algorithms were given for various classes of graphs. For the strength of graphs, Brook's-type theorem was proved in [11]. On the other hand, there are graphs with $s(G) > \chi(G)$ [8]. Some bounds for the vertex-chromatic sum of a graph were given in [23].

Similar to the sum vertex-coloring and vertex-chromatic sum of graphs, in [5, 9, 11], sum edge-coloring and edge-chromatic sum of graphs was introduced. A proper edge-coloring of a graph G is a mapping $\alpha : E(G) \to \mathbf{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. If α is a proper edge-coloring of a graph G, then $\Sigma'(G,\alpha)$ denotes the sum of the colors of the edges of G. For a graph G, define the edge-chromatic sum $\Sigma'(G)$ as follows: $\Sigma'(G) = \min_\alpha \Sigma'(G,\alpha)$, where minimum is taken among all possible proper edge-colorings of G. If α is a proper edge-coloring of a graph G and $\Sigma'(G) = \Sigma'(G,\alpha)$, then α is called a sum edge-coloring. The edge-strength of a graph G ($s'(G)$) is the minimum number of colors needed for a sum edge-coloring of G. For the edge-strength of graphs, Vizing's-type theorem was proved in [11]. In [5], Bar-Noy et al. proved that the problem of finding the edge-chromatic sum is NP-hard for multigraphs. Later, in [9], it was shown that the problem is NP-complete for bipartite graphs with maximum degree 3. Also, in [9], the authors proved that the problem can be solved in polynomial time for trees and that $s'(G) = \chi'(G)$ for bipartite graphs. In [20], Salavatipour proved that determining the edge-chromatic sum and the edge-strength are NP-complete for r-regular graphs with $r \geq 3$. Also he proved that $s'(G) = \chi'(G)$ for regular graphs. On the other hand, there are graphs with $\chi'(G) = \Delta(G)$ and $s'(G) = \Delta(G) + 1$ [11]. Recently, Cardinal et al. [7] determined the edge-strength of the multicycles.

In the present paper we give a polynomial time $\frac{11}{8}$-approximation algorithm for the edge-chromatic sum problem of r-regular graphs for $r \geq 3$. Next, we show that the problem of finding the edge-chromatic sum remains NP-complete even for some restricted class of bipartite graphs with maximum degree 3. Finally, we give upper bounds for the edge-chromatic sum of some split graphs.

2. Definitions and necessary results

A proper t-coloring is a proper edge-coloring which makes use of t different colors. If α is a proper t-coloring of G and $v \in V(G)$, then $S(v,\alpha)$ denotes set of colors appearing on edges incident to v. Let G be a graph and $R \subseteq V(G)$. A proper t-coloring of a graph G is called an R-sequential t-coloring [1, 2] if the edges incident to each vertex $v \in R$ are

colored by the colors $1, \ldots, d_G(v)$. For positive integers a and b, we denote by $[a,b]$, the set of all positive integers c with $a \leq c \leq b$. For a positive integer n, let K_n denote the complete graph on n vertices.

We will use the following four results.

Theorem 1 *[15]. If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

Theorem 2 *[24]. For every graph G,*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Theorem 3 *[25]. For the complete graph K_n with $n \geq 2$,*

$$\chi'(K_n) = \begin{cases} n-1, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 4 *[9, 11]. If G is a bipartite or a regular graph, then $s'(G) = \chi'(G)$.*

3. Edge-chromatic sums of regular graphs

In this section we consider the problem of finding the edge-chromatic sum of regular graphs. It is easy to show that the edge-chromatic sum problem of graphs G with $\Delta(G) \leq 2$ can be solved in polynomial time. On the other hand, in [19], it was proved that the problem of finding the edge-chromatic sum of an r-regular ($r \geq 3$) graph is NP-complete. Clearly, $\Sigma'(G) \geq \frac{nr(r+1)}{4}$ for any r-regular graph G with n vertices, since the sum of colors appearing on the edges incident to any vertex is at least $\frac{r(r+1)}{2}$. Moreover, it is easy to see that $\Sigma'(G) = \frac{nr(r+1)}{4}$ if and only if $\chi'(G) = r$ for any r-regular graph G with n vertices.

First we give a result on R-sequential colorings of regular graphs and then we use this result for constructing an approximation algorithm.

Theorem 5 *If G is an r-regular graph with n vertices, then G has an R-sequential $(r+1)$-coloring with $|R| \geq \left\lceil \frac{n}{r+1} \right\rceil$.*

Proof. By Theorem 2, there exists a proper $(r+1)$-coloring α of the graph G. For $i = 1, 2, \ldots, r+1$, define the set $V_\alpha(i)$ as follows:

$$V_\alpha(i) = \{v \in V(G) : i \notin S(v, \alpha)\}.$$

Clearly, for any $i', i'', 1 \leq i' < i'' \leq r+1$, we have

$$V_\alpha(i') \cap V_\alpha(i'') = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{r+1} V_\alpha(i) = V(G).$$

Hence,

$$n = |V(G)| = \left|\bigcup_{i=1}^{r+1} V_\alpha(i)\right| = \sum_{i=1}^{r+1} |V_\alpha(i)|.$$

This implies that there exists i_0, $1 \leq i_0 \leq r+1$, for which $|V_\alpha(i_0)| \geq \left\lceil \frac{n}{r+1} \right\rceil$. Let $R = V_\alpha(i_0)$.

If $i_0 = r+1$, then α is an R-sequential $(r+1)$-coloring of G; otherwise define an edge-coloring β as follows: for any $e \in E(G)$, let

$$\beta(e) = \begin{cases} \alpha(e), & \text{if } \alpha(e) \neq i_0, r+1, \\ i_0, & \text{if } \alpha(e) = r+1, \\ r+1, & \text{if } \alpha(e) = i_0. \end{cases}$$

It is easy to see that β is an R-sequential $(r+1)$-coloring of G with $|R| \geq \left\lceil \frac{n}{r+1} \right\rceil$. $\square$

Corollary 6 *If G is a cubic graph with n vertices, then G has an R-sequential 4-coloring with $|R| \geq \left\lceil \frac{n}{4} \right\rceil$.*

Note that if n is odd, then the lower bound in Theorem 5 cannot be improved, since the complete graph K_n has an R-sequential n-coloring with $|R| = 1$.

The theorem we are going to prove will be used in section 5.

Theorem 7 *For any $n \in \mathbf{N}$, we have*

$$\Sigma'(K_n) = \begin{cases} \frac{n(n^2-1)}{4}, & \text{if } n \text{ is odd}, \\ \frac{(n-1)n^2}{4}, & \text{if } n \text{ is even}. \end{cases}$$

Proof. Since for any r-regular graph G with n vertices, $\Sigma'(G) = \frac{nr(r+1)}{4}$ if and only if $\chi'(G) = r$ and, by Theorems 3 and 4, we obtain $\Sigma'(K_n) = \frac{(n-1)n^2}{4}$ if n is even.

Now let n be an odd number and $n \geq 3$. In this case by Theorems 3 and 4, we have $s'(K_n) = \chi'(K_n) = n$. It is easy to see that in any proper n-coloring of K_n the missing colors at n vertices are all distinct. Hence,

$$\Sigma'(K_n) = \frac{\frac{n^2(n+1)}{2} - \frac{n(n+1)}{2}}{2} = \frac{n(n^2-1)}{4}.$$

$\square$

In [5], it was shown that there exists a 2-approximation algorithm for the edge-chromatic sum problem on general graphs. Now we show that there exists a $\left(1 + \frac{2r}{(r+1)^2}\right)$-approximation algorithm for the edge-chromatic sum problem on r-regular graphs for $r \geq 3$. Note that $1 + \frac{2r}{(r+1)^2}$ decreases for increasing r and $\frac{11}{8}$ is its maximum value achieved for $r = 3$. Thus, we show that there is a $\frac{11}{8}$-approximation algorithm for the edge-chromatic sum problem on regular graphs.

Theorem 8 *For any $r \geq 3$, there is a polynomial time $\left(1+\frac{2r}{(r+1)^2}\right)$-approximation algorithm for the edge-chromatic sum problem on r-regular graphs.*

Proof. Let G be an r-regular graph with n vertices and m edges. Now we describe a polynomial time algorithm A for constructing a special proper $(r+1)$-coloring of G. First we construct a proper $(r+1)$-coloring α of G in $O(mn)$ time [21]. Next we recolor some edges as it is described in the proof of Theorem 5 to obtain an R-sequential $(r+1)$-coloring β of G with $|R| \geq \lceil \frac{n}{r+1} \rceil$. Clearly, we can do it in $O(m)$ time. Now, taking into account that the sum of colors appearing on the edges incident to any vertex is at most $\frac{r(r+3)}{2}$, we have

$$\begin{aligned}\Sigma'_A(G) = \Sigma'(G,\beta) &\leq \frac{\frac{r(r+1)}{2}\left\lceil \frac{n}{r+1}\right\rceil + \left(n - \left\lceil \frac{n}{r+1}\right\rceil\right)\frac{r(r+3)}{2}}{2} \leq \frac{\frac{r(r+1)}{2}\frac{n}{r+1} + \left(n - \frac{n}{r+1}\right)\frac{r(r+3)}{2}}{2} \\ &= \frac{\frac{r(r+1)}{2}\frac{n}{r+1} + \frac{nr}{r+1}\frac{r(r+3)}{2}}{2} = \frac{nr(r^2+4r+1)}{4(r+1)}.\end{aligned}$$

On the other hand, since $\Sigma'(G) \geq \frac{nr(r+1)}{4}$, we get

$$\frac{\Sigma'_A(G)}{\Sigma'(G)} \leq \frac{nr(r^2+4r+1)}{4(r+1)} \cdot \frac{4}{nr(r+1)} = \frac{r^2+4r+1}{(r+1)^2} = 1 + \frac{2r}{(r+1)^2}.$$

This shows that there exists a $\left(1+\frac{2r}{(r+1)^2}\right)$-approximation algorithm for the edge-chromatic sum problem on r-regular graphs. Moreover, we can construct the aforementioned coloring β for a regular graph in $O(mn)$ time. $\square$

4. Edge-chromatic sums of bipartite graphs

In this section we consider the problem of finding the edge-chromatic sum of bipartite graphs. Let $G = (U \cup W, E)$ be a bipartite graph with a bipartition (U, W). By $U_i \subseteq U$ and $W_i \subseteq W$, we denote sets of vertices of degree i in U and W, respectively. Define sets $V_{\geq i} \subseteq V(G)$ and $U_{\geq i} \subseteq U$ as follows: $V_{\geq i} = \{v : v \in V(G) \wedge d_G(v) \geq i\}$ and $U_{\geq i} = \{u \in V(G) : u \in U \wedge d_G(u) \geq i\}$. It was proved the following:

Theorem 9 *[1, 2, 3, 4] If $G = (U \cup W, E)$ is a bipartite graph with $d_G(u) \geq d_G(w)$ for every $uw \in E(G)$, where $u \in U$ and $w \in W$, then G has a U-sequential $\Delta(G)$-coloring.*

By this theorem, we obtain the following corollary:

Corollary 10 *If $G = (U \cup W, E)$ is a bipartite graph with $d_G(u) \geq d_G(w)$ for every $uw \in E(G)$, where $u \in U$ and $w \in W$, then a U-sequential $\Delta(G)$-coloring of G is a sum edge-coloring of G and $\Sigma'(G) = \sum_{u\in U} \frac{d_G(u)(d_G(u)+1)}{2}$.*

In [9], it was shown that the problem of finding the edge-chromatic sum of bipartite graphs G with $\Delta(G) = 3$ is NP-complete. Now we give a short proof of this fact. First we need the following

Problem 1. [2, 4, 14]

Instance: A bipartite graph $G = (U \cup W, E)$ with $\Delta(G) = 3$.

Question: Is there a U-sequential 3-coloring of G?

It was proved the following:

Theorem 11 *[2, 14] Problem 1 is NP-complete.*

Now let us consider the following

Problem 2.

Instance: A bipartite graph $G = (U \cup W, E)$ with $\Delta(G) = 3$.

Question: Is $\Sigma'(G) = \sum\limits_{i=1}^{3} i \cdot |U_{\geq i}|$?

Theorem 12 *Problem 2 is NP-complete.*

Proof. Clearly, Problem 2 belongs to NP. For the proof of the NP-completeness, we show a reduction from Problem 1 to Problem 2. We prove that a bipartite graph $G = (U \cup W, E)$ with $\Delta(G) = 3$ admits a U-sequential 3-coloring if and only if $\Sigma'(G) = \sum\limits_{i=1}^{3} i \cdot |U_{\geq i}|$. Let $G = (U \cup W, E)$ be a bipartite graph with $\Delta(G) = 3$ and α be a U-sequential 3-coloring of G. In this case the colors $1, 2, 3$ appear on the edges incident to each vertex $u \in U_3$, the colors $1, 2$ appear on the edges incident to each vertex $u \in U_2$ and the color 1 appears on the pendant edges incident to each vertex $u \in U_1$. Hence, $\Sigma'(G, \alpha) = \sum\limits_{i=1}^{3} i \cdot |U_{\geq i}|$. On the other hand, clearly, $\Sigma'(G) \geq \sum\limits_{i=1}^{3} i \cdot |U_{\geq i}|$, thus $\Sigma'(G) = \sum\limits_{i=1}^{3} i \cdot |U_{\geq i}|$.

Now suppose that $\Sigma'(G) = \sum\limits_{i=1}^{3} i \cdot |U_{\geq i}|$. By Theorems 1 and 4, there exists a proper 3-coloring β of a bipartite graph G with $\Delta(G) = 3$. This implies that the colors $1, 2, 3$ appear on the edges incident to each vertex $u \in U_3$. If the color 3 appears on the edges incident to some vertices $u \in U_2$ or the colors 2 or 3 appear on the pendant edges incident to some vertices $u \in U_1$, then it is easy to see that $\Sigma'(G, \beta) > \sum\limits_{i=1}^{3} i \cdot |U_{\geq i}|$. Hence, β is a U-sequential 3-coloring of G. $\square$

Now we prove that the problem of finding the edge-chromatic sum of bipartite graphs G with $\Delta(G) = 3$ and with additional conditions is NP-complete, too. We need the following

Problem 3. [2, 14]

Instance: A bipartite graph $G = (U \cup W, E)$ with $\Delta(G) = 3$ and $|U_i| = |W_i|$ for $i = 1, 2, 3$.

Question: Is there a $V(G)$-sequential 3-coloring of G?

It was proved the following:

Theorem 13 *[2, 14] Problem 3 is NP-complete.*

Now let us consider the following

Problem 4.

Instance: A bipartite graph $G = (U \cup W, E)$ with $\Delta(G) = 3$ and $|U_i| = |W_i|$ for $i = 1, 2, 3$.

Question: Is $\Sigma'(G) = \frac{1}{2}\sum\limits_{i=1}^{3} i \cdot |V_{\geq i}|$?

Theorem 14 *Problem 4 is NP-complete.*

Proof. Clearly, Problem 4 belongs to NP. For the proof of the NP-completeness, we show a reduction from Problem 3 to Problem 4. We prove that a bipartite graph $G = (U \cup W, E)$ with $\Delta(G) = 3$ and $|U_i| = |W_i|$ for $i = 1, 2, 3$, admits a $V(G)$-sequential 3-coloring if and only if $\Sigma'(G) = \frac{1}{2}\sum\limits_{i=1}^{3} i \cdot |V_{\geq i}|$. Let α be a $V(G)$-sequential 3-coloring of G. In this case the colors $1, 2, 3$ appear on the edges incident to each vertex $v \in V(G)$ with $d_G(v) = 3$, the colors $1, 2$ appear on the edges incident to each vertex $v \in V(G)$ with $d_G(v) = 2$ and the color 1 appears on the pendant edges incident to each vertex $v \in V(G)$ with $d_G(v) = 1$. Hence, $\Sigma'(G, \alpha) = \frac{1}{2}\sum\limits_{i=1}^{3} i \cdot |V_{\geq i}|$. On the other hand, clearly, $\Sigma'(G) \geq \frac{1}{2}\sum\limits_{i=1}^{3} i \cdot |V_{\geq i}|$, thus $\Sigma'(G) = \frac{1}{2}\sum\limits_{i=1}^{3} i \cdot |V_{\geq i}|$.

Now suppose that $\Sigma'(G) = \frac{1}{2}\sum\limits_{i=1}^{3} i \cdot |V_{\geq i}|$. By Theorems 1 and 4, there exists a proper 3-coloring β of a bipartite graph G with $\Delta(G) = 3$ and $|U_i| = |W_i|$ for $i = 1, 2, 3$. This implies that the colors $1, 2, 3$ appear on the edges incident to each vertex $v \in V(G)$ with $d_G(v) = 3$. If the color 3 appears on the edges incident to some vertices $v \in V(G)$ with $d_G(v) = 2$ or the colors 2 or 3 appear on the pendant edges incident to some vertices $v \in V(G)$ with $d_G(v) = 1$, then it is easy to see that $\Sigma'(G, \beta) > \frac{1}{2}\sum\limits_{i=1}^{3} i \cdot |V_{\geq i}|$. Hence, β is a $V(G)$-sequential 3-coloring of G. $\square$

In [19], it was proved that the problem of finding the edge-chromatic sum of bipartite graphs G with $\Delta(G) = 3$ remains NP-hard even for planar bipartite graphs.

5. Edge-chromatic sums of split graphs

In this section we consider the problem of finding the edge-chromatic sum of split graphs. A split graph is a graph whose vertices can be partitioned into a clique C and an independent set I. Let $G = (C \cup I, E)$ be a split graph, where $C = \{u_1, u_2, \ldots, u_n\}$ is clique and $I = \{v_1, v_2, \ldots, v_m\}$ is independent set. Define a number Δ_I as follows: $\Delta_I = \max_{1\leq j\leq m} d_G(v_j)$. Define subgraphs H and H' of a graph G as follows:

$$H = (C \cup I, E(G) \setminus E(G[C])) \text{ and } H' = G[C].$$

Clearly, H is a bipartite graph with a bipartition (C, I), and $d_H(u_i) = d_G(u_i) - n + 1$ for $i = 1, 2, \ldots, n$, $d_H(v_j) = d_G(v_j)$ for $j = 1, 2, \ldots, m$.

Theorem 15 *Let $G = (C \cup I, E)$ be a split graph, where $C = \{u_1, u_2, \ldots, u_n\}$ is clique and $I = \{v_1, v_2, \ldots, v_m\}$ is independent set. If $d_G(u_i) - d_G(v_j) \geq n - 1$ for every $u_i v_j \in E(G)$, then:*

(1) *if n is even, then*

$$\Sigma'(G) \leq$$
$$\min\left\{\sum_{i=1}^{n} \frac{(d_G(u_i)-n+1)(d_G(u_i)-n+2)}{2} + \frac{(2\Delta(G)-n+2)n(n-1)}{4}, \Sigma'(K_n) + \sum_{i=1}^{n} \frac{(d_G(u_i)-n+1)(d_G(u_i)+n)}{2}\right\};$$

(2) *if n is odd, then*

$$\Sigma'(G) \leq$$
$$\min\left\{\sum_{i=1}^{n} \frac{(d_G(u_i)-n+1)(d_G(u_i)-n+2)}{2} + \frac{(2\Delta(G)-n+3)n(n-1)}{4}, \Sigma'(K_n) + \sum_{i=1}^{n} \frac{(d_G(u_i)-n+1)(d_G(u_i)+n+2)}{2}\right\}.$$

Proof. For the proof, we are going to construct edge-colorings that satisfies the specified conditions.

Since $d_G(u_i) - d_G(v_j) \geq n - 1$ for every $u_i v_j \in E(G)$, we have $d_H(u_i) \geq d_H(v_j)$ for each $u_i v_j \in E(H)$. By Theorem 9, there exists a C-sequential $\Delta(H)$-coloring α of the graph H and, by Corollary 10, we obtain

$$\Sigma'(H) = \Sigma'(H, \alpha) = \sum_{i=1}^{n} \frac{d_H(u_i)(d_H(u_i)+1)}{2}.$$

Now we consider two cases.

Case 1: n is even.

In this case, by Theorem 3, we have $\chi'(H') = n - 1$. Let β be a proper edge-coloring of a graph H' with colors $\Delta(G)-n+2, \ldots, \Delta(G)$. Clearly, for each vertex u_i, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to u_i in H' is $[\Delta(G) - n + 2, \Delta(G)]$. Thus, we obtain

$$\Sigma'(G) \leq \Sigma'(H) + \frac{(2\Delta(G)-n+2)n(n-1)}{4}.$$

On the other hand, let β' be a proper edge-coloring of a graph H' with colors $1, 2, \ldots, n-1$. Clearly, for each vertex u_i, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to u_i in H' is $[1, n-1]$. Next, we define an edge-coloring γ of the graph H as follows: for every $e \in E(H)$, let $\gamma(e) = \alpha(e) + n - 1$. Thus, we obtain

$$\Sigma'(G) \leq \Sigma'(K_n) + \sum_{i=1}^{n} \frac{(d_G(u_i)-n+1)(d_G(u_i)+n)}{2}.$$

Case 2: n is odd.

In this case, by Theorem 3, we have $\chi'(H') = n$. Let β be a proper edge-coloring of a graph H' with colors $\Delta(G) - n + 2, \ldots, \Delta(G) + 1$. Without loss of generality, we may assume that for each vertex u_i, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to u_i in H' is $[\Delta(G) - n + 2, \Delta(G) + 1] \setminus \{\Delta(G) - n + 1 + i\}$. Thus, we obtain

$$\Sigma'(G) \leq \Sigma'(H) + \tfrac{(2\Delta(G)-n+3)n(n-1)}{4}.$$

On the other hand, let β' be a proper edge-coloring of a graph H' with colors $1, 2, \ldots, n$. Without loss of generality, we may assume that for each vertex u_i, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to u_i in H' is $[1, n] \setminus \{i\}$. Next, we define an edge-coloring γ of the graph H as follows: for every $e \in E(H)$, let $\gamma(e) = \alpha(e) + n$. Thus, we obtain

$$\Sigma'(G) \leq \Sigma'(K_n) + \textstyle\sum_{i=1}^{n} \tfrac{(d_G(u_i)-n+1)(d_G(u_i)+n+2)}{2}.$$

□

Theorem 16 *Let $G = (C \cup I, E)$ be a split graph, where $C = \{u_1, u_2, \ldots, u_n\}$ is clique and $I = \{v_1, v_2, \ldots, v_m\}$ is independent set. If $d_G(u_i) - d_G(v_j) \leq n - 1$ for every $u_iv_j \in E(G)$, then:*

(1) *if n is even, then*

$$\Sigma'(G) \leq \min\left\{\textstyle\sum_{j=1}^{m} \tfrac{d_G(v_j)(d_G(v_j)+1)}{2} + \tfrac{(2\Delta_I+n)n(n-1)}{4}, \Sigma'(K_n) + \textstyle\sum_{j=1}^{m} \tfrac{d_G(v_j)(d_G(v_j)+2n-1)}{2}\right\};$$

(2) *if n is odd, then*

$$\Sigma'(G) \leq \min\left\{\textstyle\sum_{j=1}^{m} \tfrac{d_G(v_j)(d_G(v_j)+1)}{2} + \tfrac{(2\Delta_I+n+1)n(n-1)}{4}, \Sigma'(K_n) + \textstyle\sum_{j=1}^{m} \tfrac{d_G(v_j)(d_G(v_j)+2n+1)}{2}\right\}.$$

Proof. For the proof, we are going to construct edge-colorings that satisfies the specified conditions.

Since $d_G(u_i) - d_G(v_j) \leq n - 1$ for every $u_iv_j \in E(G)$, we have $d_H(u_i) \leq d_H(v_j)$ for each $u_iv_j \in E(H)$. By Theorem 9, there exists an I-sequential Δ_I-coloring α of the graph H and, by Corollary 10, we obtain

$$\Sigma'(H) = \Sigma'(H, \alpha) = \textstyle\sum_{j=1}^{m} \tfrac{d_H(v_j)(d_H(v_j)+1)}{2} = \textstyle\sum_{j=1}^{m} \tfrac{d_G(v_j)(d_G(v_j)+1)}{2}.$$

Now we consider two cases.

Case 1: n is even.

In this case, by Theorem 3, we have $\chi'(H') = n - 1$. Let β be a proper edge-coloring of a graph H' with colors $\Delta_I + 1, \ldots, \Delta_I + n - 1$. Clearly, for each vertex u_i, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to u_i in H' is $[\Delta_I + 1, \Delta_I + n - 1]$. Thus, we obtain

$$\Sigma'(G) \leq \Sigma'(H) + \tfrac{(2\Delta_I+n)n(n-1)}{4}.$$

On the other hand, let β' be a proper edge-coloring of a graph H' with colors $1, 2, \ldots, n-1$. Clearly, for each vertex u_i, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to u_i in H' is $[1, n-1]$. Next, we define an edge-coloring γ of the graph H as follows: for every $e \in E(H)$, let $\gamma(e) = \alpha(e) + n - 1$. Thus, we obtain

$$\Sigma'(G) \leq \Sigma'(K_n) + \sum_{j=1}^{m} \frac{d_G(v_j)(d_G(v_j)+2n-1)}{2}.$$

Case 2: n is odd.

In this case, by Theorem 3, we have $\chi'(H') = n$. Let β be a proper edge-coloring of a graph H' with colors $\Delta_I + 1, \ldots, \Delta_I + n$. Without loss of generality, we may assume that for each vertex u_i, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to u_i in H' is $[\Delta_I + 1, \Delta_I + n] \setminus \{\Delta_I + i\}$. Thus, we obtain

$$\Sigma'(G) \leq \Sigma'(H) + \frac{(2\Delta_I+n+1)n(n-1)}{4}.$$

On the other hand, let β' be a proper edge-coloring of a graph H' with colors $1, 2, \ldots, n$. Without loss of generality, we may assume that for each vertex u_i, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to u_i in H' is $[1, n] \setminus \{i\}$. Next, we define an edge-coloring γ of the graph H as follows: for every $e \in E(H)$, let $\gamma(e) = \alpha(e) + n$. Thus, we obtain

$$\Sigma'(G) \leq \Sigma'(K_n) + \sum_{j=1}^{m} \frac{d_G(v_j)(d_G(v_j)+2n+1)}{2}.$$

□

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